Solution – Tamalin drum

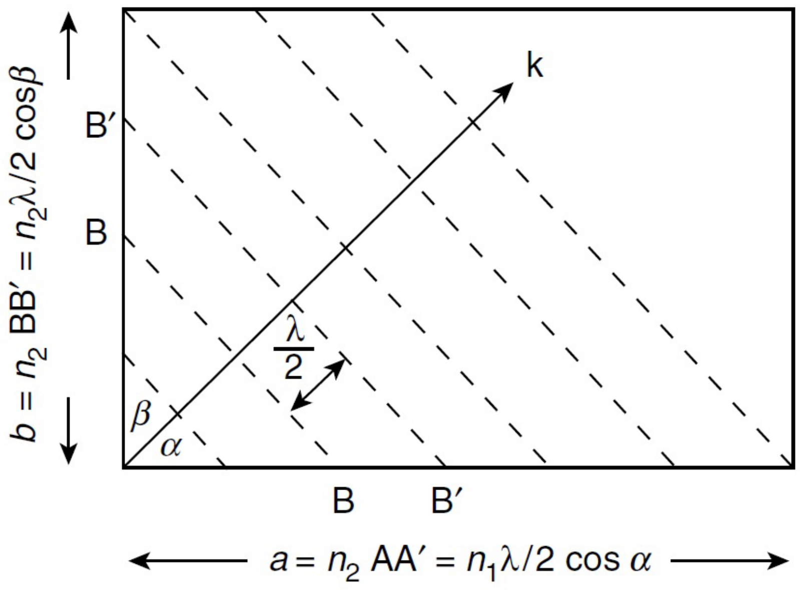

Figure 1: schematic representation rectangular membrane measuring $a = 60.0$ cm and $b = 20.0$ cm [1].

1.
Tension T

$$T = \rho_a \cdot c^2, \tag{1}$$

depends on the given surface density $\rho_a = 2.10$ kg/m^2 and the still unknown wave velocity c (m/s). We start out with the wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial^2 t} = \nabla^2 u, \tag{2}$$

or, written differently,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial^2 t} = \frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y}, \tag{3}$$

If we assume that x, y and t are independent variables, we can write the solution of the wave equation as a product of three independent solutions

$$u(x, y, t) = X(\mathrm{x}) \cdot Y(y) \cdot T(t), \tag{4}$$

where the function $T(t)$ is not to be confused with the tension in the membrane. It is a function solely depending on time. When we differentiate the chosen solution $u(x, y, y)$ twice (with respect to every parameter x, y and t) and divide by $u(x, y, t)$, we find

$$\frac{X_{xx}}{X} + \frac{Y_{yy}}{Y} = \frac{1}{c^2}\frac{T_{tt}}{T}, \tag{5}$$

or

$$\frac{X_{xx}}{X} + \frac{Y_{yy}}{Y} - \frac{1}{c^2}\frac{T_{tt}}{T} = 0. \tag{6}$$

where notation X_{xx} means we have differentiated $X(x)$ twice with respect to x, etc. Equation 6 can only yield zero for every combination (x,y,t) when the individual terms are constant. Hence we equate every term to a constant, $k_x^{\ 2}$, $k_y^{\ 2}$ and k^2

$$\underbrace{\frac{X_{xx}}{X}}_{=k_x^2} + \underbrace{\frac{Y_{yy}}{Y}}_{=k_y^2} - \underbrace{\frac{1}{c^2}\frac{T_{tt}}{T}}_{=k^2} = 0, \tag{7}$$

in other words,

$$k_x^{\ 2} + k_y^{\ 2} - k^2 = 0, \tag{8}$$

and since $X(x)$, $Y(y)$ and $T(t)$ are independent, $\frac{X_{xx}}{X}$, $\frac{Y_{yy}}{y}$ and $T(t)$ also need to equal a constant

$$\frac{X_{xx}}{X} = k_x^2 \tag{9}$$

$$\frac{Y_{yy}}{Y} = k_y^2, \tag{10}$$

and

$$\frac{1}{c^2}\frac{T_{tt}}{T} = k^2. \tag{11}$$

So now we have the following equations

$$X_{xx} = X \cdot k_x^{\ 2}, \tag{12}$$

$$Y_{yy} = Y \cdot k_y^{\ 2}, \tag{13}$$

$$T_{tt} = T \cdot c^2 k^2. \tag{14}$$

The reason we choose k^2 is that k is exactly the wave number $k = (2\pi)/\lambda$ in this case. For each of these three equations, the second derivation is directly proportional to itself, which only applies when

$$X(x) \propto e^{i(k_x \cdot x+\phi_x)}, \tag{15}$$
$$Y(y) \propto e^{i(k_y \cdot y+\phi_y)}, \tag{16}$$

and

$$T(t) \propto e^{i(ckt+\phi_t)}, \tag{17}$$

so the solution may be written as

$$u(x, y, t) = A \cdot e^{i(k_x \cdot x+\phi_x)} \cdot e^{i(k_y \cdot y+\phi_y)} \cdot e^{i(ckt+\phi_t)}. \tag{18}$$

We are interested only in the real part of $X(x)$, $Y(y)$ and $T(t)$, so we write

$$U(x, y, t) = \Re(X(x)) \cdot \Re(Y(y)) \cdot \Re(T(t)), \tag{19}$$

and therefore

$$U(x, y, t) = \cos(k_x \cdot x + \phi_x) \cdot \cos(k_y \cdot y + \phi_y) \cdot \cos(kct + \phi_t), \tag{20}$$

where $U(x, y, t)$ is also a solution of the wave equation. The phases ϕ_x, ϕ_y and ϕ_t were added every time to enable us to meet the boundary conditions:

1. $U(0, y, t) = 0$ for all $(y, t) \rightarrow$ yielding $\phi_x = \pi/2$,
2. $U(x, 0, t) = 0$ for all (x, t) ! yielding $\phi_y = \pi/2$,
3. $U(a, y, t) = 0$ for all (y, t) ! yielding $k_x = (n_x\pi)/a$,
4. $U(x, b, t) = 0$ for all (x, t) ! yielding $k_y = (n_y\pi)/b$,

with n_x and n_y integers. We simply take phase ϕ_t as zero; after all, we can start anywhere we want to. Since we want k to be the exact wave number $(2\pi)/\lambda$ we write

$$k^2 = k_x^2 + k_y^2 = \frac{n_x^2\pi^2}{a^2} + \frac{n_y^2\pi^2}{b^2} = \frac{4\pi^2}{\lambda^2}, \tag{21}$$

which yields

$$\frac{2}{\lambda} = \sqrt{\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}}. \tag{22}$$

Hence the potential frequencies $f = c/\lambda$ meeting the boundary conditions are

$$f = \frac{c}{2}\sqrt{\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2}}, \tag{23}$$

with n_x and n_y integers, 1,2,3,...

Now that we know this, we can answer the questions. Substituting a frequency of 100 Hz, $n_x = n_y = 1$, and the measurements a = 60.0 cm and b = 20.0 cm in equation 23 yields wave velocity c = 37.9(473) m/s.

1.
So tension T in the membrane has to be

$$T = \rho_a \cdot c^2 = 3{,}02 \cdot 10^3 \text{ N/m}. \tag{24}$$

2.
Modi (3,3) and (9,1) have the exact same frequencies for all membranes for which a/b = 3(,00..) is true. This is obvious from equation 23. This is an interesting phenomenon; two separate states of the membrane have the exact same frequency nevertheless. This phenomenon occurs more in nature. These energy states of atoms are called ‘degenerate’, in which case we also have two electron configurations with the same energy level. Only after we place the atoms in a magnetic field we can see that the energy levels are split. (Zeeman Effect).

Literature
1. *The Physics of Vibrations and Waves*, Sixth Edition, H. J. Pain, Department of Physics, Imperial College of Science and Technology, London, UK, John Wiley & Sons, ISBN 0 470 01295 6.